

THE MATRIX LIE ALGEBRA ON A ONE-STEP LADDER IS ZERO PRODUCT DETERMINED

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ABSTRACT. The class of *matrix algebras on a ladder* \mathcal{L} generalizes the class of block upper triangular matrix algebras. It was previously shown that the matrix algebra on a ladder \mathcal{L} is zero product determined under matrix multiplication. In this article, we show that the matrix algebra on a one-step ladder is zero product determined under the Lie bracket.

1. INTRODUCTION

In [3], the authors defined a class of matrix algebras, the *ladder matrix algebras*, that generalizes the class of block upper triangular matrix algebras. They introduce the notion of an upper triangular k -step ladder as a method of parameterizing and indexing these algebras. Certain one-step ladder matrix algebras arise as ideals of derivation algebras of parabolic subalgebras of reductive Lie algebras, which provided the motivation for their study [2].

While these terms are made precise in Section 2, the concepts are perhaps best illustrated with an example. Let $\mathcal{L} = \{(3, 2), (6, 5)\}$. \mathcal{L} is then a 2-step upper triangular ladder on 6. The ladder matrix algebra on \mathcal{L} is the subalgebra

$$M_{\mathcal{L}} = \left\{ \begin{pmatrix} 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

of $M^{n \times n}$.

An algebra $(A, *)$ is *zero product determined* if each bilinear map φ on $A \times A$ that preserves zero products necessarily factors as a linear map f on A^2 composed with the algebra multiplication $*$ so that $\varphi(x, y) = f(x * y)$. The notion is motivated by the linear preserver problem in operator theory and has recently become a topic of considerable research [1]. It was previously shown that the ladder matrix algebras are zero product determined when $*$ is matrix multiplication [3]. The purpose of this paper is to show that a one-step ladder matrix algebra is zero product determined when $*$ is the Lie bracket $[x, y] = xy - yx$.

Previous work on zero product determined algebras has also considered the case where $*$ is the Jordan product $x \circ y = xy + yx$ [1]. Extending the present results on ladder matrix algebras to this case, and to the k -step case for both the Lie bracket and the Jordan product, remains a topic of interest to the author.

2. PRELIMINARIES

Let F be a field. Let n be a positive integer. Let $M_F^{n \times n}$ denote the space of n -by- n matrices with entries in F . Let $e_{i,j}$ denote the matrix whose entry in the i th row j th column is 1_F , and whose other entries are 0_F .

Definition 1. A k -step ladder on n is a set of pairs of positive integers

$$\mathcal{L} = \{(i_1, j_1), \dots, (i_k, j_k)\}$$

with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $1 \leq j_1 < j_2 < \dots < j_k \leq n$. Each pair (i_t, j_t) is called a *step* of \mathcal{L} .

Definition 2. The ladder matrices on \mathcal{L} is the subspace

$$M_{\mathcal{L}} = \text{Span} \bigcup_{t=1}^k \{e_{i,j} \mid 1 \leq i \leq i_t \text{ and } j_t \leq j \leq n\}.$$

Definition 3. A ladder \mathcal{L} is called *upper triangular* if $i_t < j_{t+1}$ for $t = 1, 2, \dots, k-1$.

Theorem 4 ([3]). Let \mathcal{L} be a ladder on n . $M_{\mathcal{L}}$ is closed under matrix multiplication (and subsequently under the Lie bracket) if and only if \mathcal{L} is upper triangular.

We remind the reader that if $x, y \in M_F^{n \times n}$, then the Lie bracket of x and y , denoted $[x, y]$, is the matrix $xy - yx$. A subspace of $M_F^{n \times n}$ closed under $[\cdot, \cdot]$ is termed a *Lie algebra*.

In light of Theorem 4, whenever \mathcal{L} is upper triangular we will call $M_{\mathcal{L}}$ the *matrix algebra on \mathcal{L}* in case we are considering $M_{\mathcal{L}}$ as an algebra under matrix multiplication or the *matrix Lie algebra on \mathcal{L}* in case we are considering $M_{\mathcal{L}}$ as an algebra under the Lie bracket.

The following proposition establishes that the class of block upper triangular matrix algebras is a subclass of the class of ladder matrix algebras.

Proposition 5. Let $\mathfrak{q} \subseteq M_F^{n \times n}$ be a block upper triangular matrix algebra (res. Lie algebra). There is an upper triangular ladder \mathcal{L} such that $\mathfrak{q} = M_{\mathcal{L}}$.

Proof. Block upper triangular matrix algebras (res. Lie algebras) correspond with partitions of n [5]. Let $\pi = (n_1, n_2, \dots, n_k)$ be the partition of n corresponding to \mathfrak{q} . Let

$$\mathcal{L} = \left\{ \left(\sum_{i=1}^t n_i, 1 + \sum_{i=1}^{t-1} n_i \right) \mid 1 \leq t \leq k \right\}$$

where $\sum_{i=1}^0 n_i$ should be understood to be 0. \mathcal{L} is upper triangular by construction, and furthermore is constructed so that $\mathfrak{q} = M_{\mathcal{L}}$. \square

Stated perhaps more clearly, the block upper triangular matrix algebras are precisely the ladder matrix algebras where $j_{t+1} = i_t + 1$ for $t = 1, 2, \dots, k-1$.

Definition 6. An algebra over F is a pair (A, μ) where A is a vector space over F and $\mu : A \otimes A \rightarrow A$ is an F -linear map. The image of μ is denoted by A^2 .

This definition of algebra does not assume that the multiplication map μ is associative. This definition is chosen because it is agnostic to whether we are considering $M_{\mathcal{L}}$ as an associative algebra under $\mu : x \otimes y \mapsto xy$ or as a Lie algebra under $\mu : x \otimes y \mapsto [x, y]$.

Definition 7. An algebra is called *zero product determined* if for each F -linear map $\varphi : A \otimes A \rightarrow X$ (where X is an arbitrary vector space over F) the condition

$$(1) \quad \forall x, y \in A, \varphi(x \otimes y) = 0 \text{ whenever } \mu(x \otimes y) = 0$$

ensures that φ factors through μ .

$$\begin{array}{ccc} A \otimes A & & \\ \mu \downarrow & \searrow \varphi & \\ A^2 & \xrightarrow{f} & X \end{array}$$

A linear map satisfying condition 1 is said to *preserve zero products*. By φ *factors through* μ it is meant that there is a linear map $f : A^2 \rightarrow X$ such that $\varphi = f \circ \mu$, as illustrated above. If φ factors through μ , then condition 1 holds trivially. We note that in case (A, μ) is zero product determined and $\varphi : A \otimes A \rightarrow X$ preserves zero products, then the map f such that $\varphi = f \circ \mu$ is uniquely determined.

The notion of a zero-product determined algebra was introduced by Matej Brešar, Mateja Grašič, and Juana Sánchez Ortega in [1] to further the study of near-homomorphisms on Banach algebras. We present below the results of interest to us in this paper.

Theorem 8 ([1]). $M_F^{n \times n}$ considered as an algebra under either matrix multiplication or the Lie bracket is zero product determined.

Theorem 9 ([4]). The classical Lie algebras are zero product determined.

Theorem 10 ([6]). The simple Lie algebras over \mathbb{C} and their parabolic subalgebras are zero product determined.

Theorem 11 ([3]). An abelian Lie algebra is zero product determined.

Theorem 12 ([3]). If \mathcal{L} is upper triangular, then $M_{\mathcal{L}}$ under matrix multiplication is zero product determined.

Recall that $A \otimes A = \text{Span}\{x \otimes y | x, y \in A\}$. Members of $A \otimes A$ of the form $x \otimes y$ with $x, y \in A$ are called *rank-one tensors*. We will make extensive use of the following theorem.

Theorem 13 ([3]). An algebra (A, μ) is zero product determined if and only if $\text{Ker } \mu$ is generated by rank-one tensors.

We note that while $A \otimes A$ is generated by rank-one tensors by definition, an arbitrary subspace of $A \otimes A$ need not be generated by the rank-one tensors it contains.

3. MAIN RESULT

We state and prove our main result.

Proposition 14. Let \mathcal{L} be a 1-step ladder on n . The ladder matrix Lie algebra $M_{\mathcal{L}}$ is zero product determined.

Proof. Let $\mathcal{L} = \{(i_1, j_1)\}$. If $i_1 < j_1$, then $M_{\mathcal{L}}$ is abelian and is zero product determined by Theorem 11. We assume without loss of generality that $i_1 \geq j_1$.

Let $\mu : \sum_i x_i \otimes y_i \mapsto \sum_i [x_i, y_i]$. In light of Theorem 13, our task is to construct a basis of $\text{Ker } \mu$ consisting of elements of $M_{\mathcal{L}} \otimes M_{\mathcal{L}}$ of the form $x \otimes y$ with $x, y \in M_{\mathcal{L}}$.

We partition $M_{\mathcal{L}}$ into blocks of size $n_1 = j_1 - 1 \geq 0$, $n_2 = i_1 - j_1 + 1 > 0$, and $n_3 = n - i_1 \geq 0$ so that $n_1 + n_2 + n_3 = n$. Under this block scheme, $M_{\mathcal{L}}$ has the form

$$M_{\mathcal{L}} = \begin{matrix} & \begin{matrix} n_1 & n_2 & n_3 \end{matrix} \\ \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix} & \begin{pmatrix} 0 & \mathfrak{l} & \mathfrak{a} \\ 0 & \mathfrak{h} & \mathfrak{r} \\ 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

or in case $n_1 = 0$

$$M_{\mathcal{L}} = \begin{matrix} & \begin{matrix} n_2 & n_3 \end{matrix} \\ \begin{matrix} n_2 \\ n_3 \end{matrix} & \begin{pmatrix} \mathfrak{h} & \mathfrak{r} \\ 0 & 0 \end{pmatrix} \end{matrix},$$

or in case $n_3 = 0$

$$M_{\mathcal{L}} = \begin{matrix} & \begin{matrix} n_1 & n_2 \end{matrix} \\ \begin{matrix} n_1 \\ n_2 \end{matrix} & \begin{pmatrix} 0 & \mathfrak{l} \\ 0 & \mathfrak{h} \end{pmatrix} \end{matrix},$$

where each of \mathfrak{h} , \mathfrak{l} , \mathfrak{r} , and \mathfrak{a} is a subalgebra consisting of the full matrix subspace of the appropriate size. All three cases are treated simultaneously by the below argument.

$M_{\mathcal{L}}$ admits the structural decomposition

$$M_{\mathcal{L}} = \mathfrak{h} \ltimes ((\mathfrak{l} \dot{+} \mathfrak{r}) \ltimes \mathfrak{a})$$

obeying multiplication containment relations below.

$[\cdot, \cdot]$	\mathfrak{h}	\mathfrak{l}	\mathfrak{r}	\mathfrak{a}
\mathfrak{h}	\mathfrak{h}	\mathfrak{l}	\mathfrak{r}	0
\mathfrak{l}	\mathfrak{l}	0	\mathfrak{a}	0
\mathfrak{r}	\mathfrak{r}	\mathfrak{a}	0	0
\mathfrak{a}	0	0	0	0

(where $\mathfrak{l} = \mathfrak{a} = 0$ in case $n_1 = 0$ and $\mathfrak{r} = \mathfrak{a} = 0$ in case $n_3 = 0$.)

We require the dimension of $\text{Ker } \mu$.

We see that for $h \in \mathfrak{h}$ and $r \in \mathfrak{r}$ we have $[h, r] = hr$, since $rh = 0$, and similarly with $l \in \mathfrak{l}$ we have $[h, l] = -lh$. Thus $[\mathfrak{h}, \mathfrak{r}] = \mathfrak{r}$ and $[\mathfrak{h}, \mathfrak{l}] = \mathfrak{l}$. Furthermore, for $l \in \mathfrak{l}$ and $r \in \mathfrak{r}$, we have $[l, r] = lr$, whereby $[\mathfrak{l}, \mathfrak{r}] = \mathfrak{a}$. Finally, $[\mathfrak{h}, \mathfrak{h}]$ produces only the traceless matrices, thus $\dim[\mathfrak{h}, \mathfrak{h}] = \dim \mathfrak{h} - 1$.

In light of these observations, we find that $\text{Ker } \mu$ has dimension

$$\begin{aligned} & n_1^2 n_2^2 + 2n_1^2 n_2 n_3 + n_1^2 n_3^2 + 2n_1 n_2^3 + 4n_1 n_2^2 n_3 + 2n_1 n_2 n_3^2 \\ & - n_1 n_2 - n_1 n_3 + n_2^4 + 2n_2^3 n_3 + n_2^2 n_3^2 - n_2^2 - n_2 n_3 + 1. \end{aligned}$$

Each pairing of subspaces that is killed by the bracket yields its full basis of rank-one tensors to $\text{Ker } \mu$. We have:

Subspace pair	Rank-one tensors contributed
$\mu(\mathfrak{h} \otimes \mathfrak{a}) = 0 = \mu(\mathfrak{a} \otimes \mathfrak{h})$	$2n_1 n_2^2 n_3$
$\mu(\mathfrak{l} \otimes \mathfrak{a}) = 0 = \mu(\mathfrak{a} \otimes \mathfrak{l})$	$2n_1^2 n_2 n_3$
$\mu(\mathfrak{r} \otimes \mathfrak{a}) = 0 = \mu(\mathfrak{a} \otimes \mathfrak{r})$	$2n_1 n_2 n_3^2$
$\mu(\mathfrak{a} \otimes \mathfrak{a}) = 0$	$n_1^2 n_3^2$
$\mu(\mathfrak{l} \otimes \mathfrak{l}) = 0$	$n_1^2 n_2^2$
$\mu(\mathfrak{r} \otimes \mathfrak{r}) = 0$	$n_2^2 n_3^2$

Further, \mathfrak{h} is isomorphic to $M_F^{n_2 \times n_2}$, which is zero product determined as a Lie algebra by Theorem 8. By Theorem 13 there are $n_2^4 - n_2^2 + 1$ rank-one tensors in $\mathfrak{h} \otimes \mathfrak{h}$ that μ kills. The above listed rank-one tensors in $\text{Ker } \mu$ are linearly independent by construction from block pairings. This leaves

$$2n_1n_2^3 + 2n_2^3n_3 + 2n_1n_2^2n_3 - n_1n_2 - n_1n_3 - n_2n_3$$

rank-one tensors in $\text{Ker } \mu$ we have left to construct.

We examine $\mathfrak{h} \otimes \mathfrak{r}$, $\mathfrak{r} \otimes \mathfrak{h}$, and $(\mathfrak{h} + \mathfrak{r}) \otimes (\mathfrak{h} + \mathfrak{r})$. We will find that these subspaces contribute $2n_2^3n_3 - n_2n_3$ tensors to our basis.

Consider the $2n_2^3n_3 - 2n_2^2n_3$ tensors

$$T_{i,j,l,q} = e_{i,j} \otimes e_{l,q} \in \mathfrak{h} \otimes \mathfrak{r}$$

and

$$T^{i,j,l,q} = e_{l,q} \otimes e_{i,j} \in \mathfrak{r} \otimes \mathfrak{h}$$

for $i, j, l \in (n_1, n_1 + n_2]$ and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$ with $j \neq l$.

Additionally, we have $2n_2^2n_3 - 2n_2n_3$ tensors

$$S_{i,j,q} = (e_{i,j} - e_{i,j+1}) \otimes (e_{j,q} + e_{j+1,q}) \in \mathfrak{h} \otimes \mathfrak{r}$$

and

$$S^{i,j,q} = (e_{j,q} + e_{j+1,q}) \otimes (e_{i,j} - e_{i,j+1}) \in \mathfrak{r} \otimes \mathfrak{h}$$

with $i \in (n_1, n_1 + n_2]$, $j \in (n_1, n_1 + n_2 - 1]$, and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$.

Finally, we have n_2n_3 tensors of the form

$$R(i, q) = (e_{i,i} + e_{i,q}) \otimes (e_{i,i} + e_{i,q}) \in (\mathfrak{h} + \mathfrak{r}) \otimes (\mathfrak{h} + \mathfrak{r})$$

for $i \in (n_1, n_1 + n_2]$ and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$, giving the desired $2n_2^3n_3 - n_2n_3$ rank-one tensors. By applying $\mu(x \otimes y) = [x, y]$, we see that each tensor above is in $\text{Ker } \mu$. We must show that these tensors are linearly independent.

Expanding $S_{i,j,q}$ we see that

$$S_{i,j,q} = \underbrace{e_{i,j} \otimes e_{j,q} - e_{i,j+1} \otimes e_{j+1,q}}_{\notin \text{Span}\{T_{i,j,l,q}\}} + \underbrace{e_{i,j} \otimes e_{j+1,q} - e_{i,j+1} \otimes e_{j,q}}_{\in \text{Span}\{T_{i,j,l,q}\}}$$

is not in the span of the $T_{i,j,l,q}$ tensors. A similar observation shows that $S^{i,j,q}$ is not in the span of the $T^{i,j,l,q}$ tensors.

Expanding $R(i, q)$ we have

$$R(i, q) = \underbrace{e_{i,i} \otimes e_{i,i}}_{\in \mathfrak{h} \otimes \mathfrak{h}} + \underbrace{e_{i,q} \otimes e_{i,q}}_{\in \mathfrak{r} \otimes \mathfrak{r}} + \underbrace{e_{i,i} \otimes e_{i,q} + e_{i,q} \otimes e_{i,i}}_{\in \mathfrak{h} \otimes \mathfrak{r} + \mathfrak{r} \otimes \mathfrak{h}}.$$

Since $e_{i,i} \otimes e_{i,i}$ and $e_{i,q} \otimes e_{i,q}$ are in $\mathfrak{h} \otimes \mathfrak{h}$ and $\mathfrak{r} \otimes \mathfrak{r}$, respectively, and since tensors from those blocks have been accounted for above, we may subtract those terms, leaving $R'(i, q) = e_{i,i} \otimes e_{i,q} + e_{i,q} \otimes e_{i,i}$. $R'(i, q)$ is not in the span of $\{T_{i,j,l,q}, T^{i,j,l,q}\}$ since we require $j \neq l$ in $T_{i,j,l,q}$ and $T^{i,j,l,q}$.

Now, consider $S_{i,i,q} + S^{i,i,q}$ where $i < n_1 + n_2$ ($R(i, j)$ is linearly independent of the $S_{i,j,q}$ and $S^{i,j,q}$ tensors in case $i = n_1 + n_2$, since we require $j \leq n_1 + n_2 - 1$ in

$S_{i,j,q}$ and $S^{i,j,q}$). We have

$$S_{i,i,q} + S^{i,i,q} = \underbrace{e_{i,i} \otimes e_{i,q} + e_{i,q} \otimes e_{i,i}}_{=R'(i,q)} + T - (e_{i,i+1} \otimes e_{i+1,q} + e_{i+1,q} \otimes e_{i,i+1})$$

with $T \in \text{Span}\{T_{i,j,l,q}, T^{i,j,l,q}\}$, so we have

$$R'(i,q) = S_{i,i,q} + S^{i,i,q} - T + e_{i,i+1} \otimes e_{i+1,q} + e_{i+1,q} \otimes e_{i,i+1}.$$

Write $R''(i,q) = e_{i,i+1} \otimes e_{i+1,q} + e_{i+1,q} \otimes e_{i,i+1}$. Now, if $i = n_1 + n_2 - 1$ we are done (as above). If $i < n_1 + n_2 - 1$ we may reduce $R''(i,q)$ using the same method just employed, and so by induction we are done. That is to say that $T_{i,j,l,q}$, $T^{i,j,l,q}$, $S_{i,j,q}$, $S^{i,j,q}$, and $R(i,j)$ are linearly independent.

Next, we examine $\mathfrak{h} \otimes \mathfrak{l}$, $\mathfrak{l} \otimes \mathfrak{h}$, and $(\mathfrak{h} \dot{+} \mathfrak{l}) \otimes (\mathfrak{h} \dot{+} \mathfrak{l})$. The consideration of these subspaces is symmetric with the subspaces considered above, and so we will find that these subspaces contribute $2n_1n_2^3 - n_1n_2$ tensors to our basis of $\text{Ker } \mu$.

Finally, we examine $\mathfrak{l} \otimes \mathfrak{r}$, $\mathfrak{r} \otimes \mathfrak{l}$, and $(\mathfrak{l} \dot{+} \mathfrak{r}) \otimes (\mathfrak{l} \dot{+} \mathfrak{r})$. We proceed similarly to the discussion of \mathfrak{h} and \mathfrak{r} above, and we will find that \mathfrak{l} and \mathfrak{r} contribute the remaining $2n_1n_2^2n_3 - n_1n_3$ rank-one tensors needed to span $\text{Ker } \mu$.

Consider the $2n_1n_2^2n_3 - 2n_1n_2n_3$ tensors

$$U_{i,j,l,q} = e_{i,j} \otimes e_{l,q} \in \mathfrak{l} \otimes \mathfrak{r}$$

and

$$U^{i,j,l,q} = e_{l,q} \otimes e_{i,j} \in \mathfrak{r} \otimes \mathfrak{l}$$

for $i \in (0, n_1]$, $j, l \in (n_1, n_1 + n_2]$, and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$ with $j \neq l$.

Additionally, we have $2n_1n_2n_3 - 2n_1n_3$ tensors

$$V_{i,j,q} = (e_{i,j} - e_{i,j+1}) \otimes (e_{j,q} + e_{j+1,q}) \in \mathfrak{l} \otimes \mathfrak{r}$$

and

$$V^{i,j,q} = (e_{j,q} + e_{j+1,q}) \otimes (e_{i,j} - e_{i,j+1}) \in \mathfrak{r} \otimes \mathfrak{l}$$

with $i \in (0, n_1]$, $j \in (n_1, n_1 + n_2 - 1]$, and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$.

Finally, we have n_1n_3 tensors of the form

$$W(i,q) = (e_{i,n_1+n_2} + e_{n_1+n_2,q}) \otimes (e_{i,n_1+n_2} + e_{n_1+n_2,q}) \in (\mathfrak{l} \dot{+} \mathfrak{r}) \otimes (\mathfrak{l} \dot{+} \mathfrak{r})$$

for $i \in (0, n_1]$ and $q \in (n_1 + n_2, n_1 + n_2 + n_3]$, giving the remaining $2n_1n_2^2n_3 - n_1n_3$ rank-one tensors. Again, the above tensors were chosen so that applying $\mu(x \otimes y) = [x, y]$ results in 0. Below we verify that they are linearly independent.

Expanding $V_{i,j,q}$ we see that

$$V_{i,j,q} = \underbrace{e_{i,j} \otimes e_{j,q} - e_{i,j+1} \otimes e_{j+1,q}}_{\notin \text{Span}\{U_{i,j,l,q}\}} + \underbrace{e_{i,j} \otimes e_{j+1,q} - e_{i,j+1} \otimes e_{j,q}}_{\in \text{Span}\{U_{i,j,l,q}\}}$$

is not in the span of the $U_{i,j,l,q}$ tensors. A similar observation shows that $V^{i,j,q}$ is not in the span of the $U^{i,j,l,q}$ tensors.

Expanding $W(i, q)$ we have

$$\begin{aligned} W(i, q) = & \underbrace{e_{i, n_1+n_2} \otimes e_{i, n_1+n_2}}_{\in \mathfrak{l} \otimes \mathfrak{l}} + \underbrace{e_{n_1+n_2, q} \otimes e_{n_1+n_2, q}}_{\in \mathfrak{r} \otimes \mathfrak{r}} \\ & + \underbrace{e_{i, n_1+n_2} \otimes e_{n_1+n_2, q} + e_{n_1+n_2, q} \otimes e_{i, n_1+n_2}}_{\in \mathfrak{l} \otimes \mathfrak{r} + \mathfrak{r} \otimes \mathfrak{l}}. \end{aligned}$$

$\mathfrak{l} \otimes \mathfrak{l}$ and $\mathfrak{r} \otimes \mathfrak{r}$ are accounted for above, so we may subtract their terms, leaving

$$W'(i, q) = e_{i, n_1+n_2} \otimes e_{n_1+n_2, q} + e_{n_1+n_2, q} \otimes e_{i, n_1+n_2}.$$

$W'(i, q)$ is not in the span of $\{U_{i, j, l, q}, U^{i, j, l, q}\}$ since we require $j \neq l$ in $U_{i, j, l, q}$ and $U^{i, j, l, q}$. We also see immediately that $W'(i, q)$ is not in the span of $\{V_{i, j, q}, V^{i, j, q}\}$ since we require $j < n_1 + n_2$ in $V_{i, j, q}$ and $V^{i, j, q}$. Thus we have that $U_{i, j, l, q}$, $U^{i, j, l, q}$, $V_{i, j, q}$, $V^{i, j, q}$, and $W(i, j)$ are linearly independent.

Having explicitly constructed a basis for $\text{Ker } \mu$ consisting of rank-one tensors, the proof is complete. \square

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